CORRIGENDUM TO OUR PAPER: BIRATIONAL TRANSFORMATIONS OF WEIGHTED GRAPHS

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Abstract. We give a corrected version of Corollary 3.33 in [FKZ_1].

In our paper [FKZ_1] on birational transformations of weighted graph, we showed that every weighted graph Γ is birationally equivalent to a standard one.[1] Concerning the uniqueness of this standard form, we claimed in Corollary 3.33 that *a non-circular standard weighted graph is unique in its birational equivalence class up to reversion of its linear segments.*

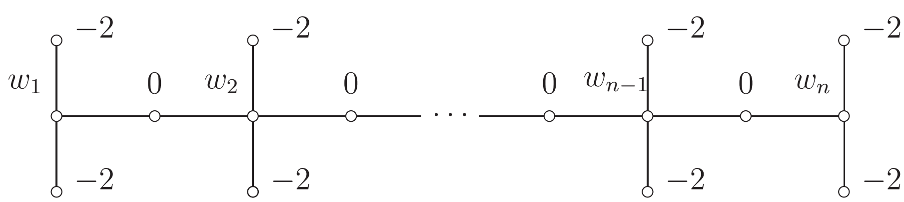
However in this form the corollary is not true as was pointed out to us by Karol Palka. The correct version is as follows.

Corollary 3.33. *Every non-circular standard weighted graph is unique in its birational equivalence class up to reversions of its linear segments* ***and changing the weights of its branching vertices****. Similarly, a circular standard graph is unique in its birational equivalence class up to a cyclic permutation of its nonzero weights and reversion.*

In the subsequent papers [FKZ_2]-[FKZ_4] we applied this result only to linear or circular graphs. The latter graphs have no branching vertices. Hence we used a correct version, and so this does not affect any of our subsequent results.

To demonstrate that this correction is needed, following a suggestion of Karol Palka let us consider the weighted tree

Γ :



Performing inner elementary transformations[2] at zero vertices, one can change the weights $w_1, \ldots, w_n$ of the branching vertices and so replace them by the new weights $w'_1 = \ldots w'_{n-1} = 0$ and $w'_n = \sum_{i=1}^n w_i$, without changing the birational class of Γ. Thus the correction above is indeed necessary.

It is possible to describe completely the possible weights of branching points in the birational equivalence class of a standard graph. For this we let B denote the set of branching points of Γ. Moreover let Γ_0 be the subgraph of Γ, which is the union of B and all linear segments of type $[[0_{2k+1}]]$ of Γ including the edges that join these segments with B. For instance, if Γ has no segments of type $[[0_{2k+1}]]$ then $\Gamma_0 = B$.

[1] See §2.1 and §2.4 in *loc.cit.* for the definitions of birational transformations, standard weighted graphs, and linear and circular segments. A standard graph is supposed to be connected.

[2] See [FKZ_1, §2.3].

With these notations the following more precise form of Corollary 3.33 holds.

Corollary 3.33′. *Let Γ be a non-circular standard weighted graph. Then Γ is unique in its birational equivalence class up to reversion of its linear segments and the change of weights of branching points described by the following procedure:*

(1) If a connected component G of Γ_0 contains an end-vertex of Γ [3] *then the weights of points of $G \cap B$ can be chosen arbitrarily;*
(2) If G does not contain any end-vertex of Γ then the weights of the points of $G \cap B$ can be chosen arbitrarily modulo preservation of the sum of weights at these branching points.

In particular, we can change the weights of Γ in such a way that for every connected component G of Γ_0 the new weights of points of $G \cap B$ are all zero in case (1), and are all zero with one exception in case (2), with the new weight in this exceptional position equal to the sum of the weights of the points of $G \cap B$.

For the proof we need a few preparations. Let us recall first the following results.

(i) Any birational transformation of one standard graph into another one is composed of a sequence of elementary transformations ([FKZ$_1$, Theorem 3.1]). In particular, it identifies the branching vertices of both graphs, and transforms linear segments of the first graph into linear segments of the second one.
(ii) A standard linear segment is unique in its birational equivalence class up to reversion ([FKZ$_1$, Corollary 3.32]). The reversion of a standard linear segment L of Γ leaves unchanged the weighted graph $\Gamma \ominus L$ ([FKZ$_1$, Lemma 2.12]).

Let $\sigma : \Gamma \dashrightarrow \Gamma'$ be a birational transformation of weighted graphs and let L be a segment of Γ. We call σ an *L-transformation* if it is composed of blowups and blowdowns of L and their subsequent total transforms. Thus σ induces a bijection $\Gamma \ominus L \to \Gamma' \ominus L'$ that respects the weights of all vertices except possibly the branching points that are adjacent to L and L', respectively. We need the following lemma.

Lemma 1. *(a) Every birational transformation $\sigma : \Gamma \dashrightarrow \Gamma'$ of standard graphs can be written as a composition of L-transformations, where L runs through the segments of Γ.*

(b) Every birational transformation $\sigma : \Gamma \dashrightarrow \Gamma'$ of standard graphs admits a domination

$$\begin{tikzcd} & \Delta \arrow[dl] \arrow[dr] & \\ \Gamma \arrow[rr, dashed] & & \Gamma' \end{tikzcd}$$

such that Δ, Γ and Γ' have the same branching points. In particular, if L is a segment of Γ and L' is the corresponding segment of Γ' then the induced birational transformation $\sigma|L : L \dashrightarrow L'$ can be dominated by a linear graph.

Proof. By (i) σ can be written as a composition of elementary transformations $\sigma_1 \cdot \ldots \cdot \sigma_n$. Every such elementary transformation takes place at some segment L of Γ. Obviously elementary transformations taking place at different segments commute. Thus if σ_L is the product of all σ_i taking place at L then σ is the product of all σ_L. This proves (a).

[3] I.e., a vertex of Γ of degree 1.

The first part of (b) follows from Lemma 3.8 in [FKZ_1] while the second is immediate from the first one. □

Proof of Corollary 3.33'. Let $\sigma : \Gamma \dashrightarrow \Gamma'$ be a birational transformation of standard graphs. By Lemma 1(b) they can be dominated by a graph Δ such that Δ, Γ and Γ' have the same branching points. Moreover by Lemma 1(a) σ can be decomposed as a product $\sigma = \prod_L \sigma_L$, where σ_L is an L-transformation.

Let us first describe the effect of σ_L on the weights of the branching points of Γ.

(a) If a branching point is not adjacent to L then its weight remains unchanged under σ_L.

(b) Assume that $L = [[0_{2k}, w_1, \ldots, w_n]]$ with $n \geq 0$ and $w_i \leq -2$ for all i. According to Proposition 3.4 in [FKZ_1] the induced birational transformation $\sigma|L = \sigma_L|L : L \dashrightarrow L'$ is either the reversion or the identity. As observed in (ii), in both cases σ_L does not affect the weights of branching points.

(c) If $L = [[0_{2k+1}]]$ then according to Proposition 3.7 in [FKZ_1] $\sigma_L|L = \tau^s$ for some $s \in \mathbb{Z}$, where τ denotes the left move (see Definition 3.5 in *loc.cit.*). Comparing with Lemma 2.12(c) in *loc.cit.*, σ_L will increase the weight of one of the branching points adjacent to L by s, while it decreases the weight of the other one (if existent) by s.

In any case, for every connected component G of Γ_0 the sum of the weights of points in $G \cap B$ is invariant under σ_L and hence also under σ.

Finally, applying (c) repeatedly we can move the weight of any vertex v in $G \cap B$ to any other vertex v' in $G \cap B$ along a path connecting them in G. To annihilate all the weights but one, we proceed as follows. We choose a rooted subtree of G containing all the vertices of $G \cap B$ (and the root is one of them), and move recursively the weights to the root starting from the vertices in $G \cap B$ with maximal distance from the root. In the case that G contains an end vertex of Γ, choosing for the root the vertex in $G \cap B$ closest to the end vertex, again by (c) we finish the recursive procedure by annihilating the weight of the root.[4] This proves Corollary 3.33′. □

Acknowledgments. It is pleasure for us to thank Karol Palka, who attracted our attention to the flaw in [FKZ_1]. We also grateful to Daniel Daigle, who indicated in a letter to the authors that in the new formulation, Corollary 3.33′ (for the particular case where Γ is a forest) is essentially equivalent to the classification result obtained in [Da_2]. The case of a linear graph is contained also in [Da_1]. The relation to the earlier classification due to Danilov and Gizatullin [DG] is explained in [FKZ_1].

References

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[4]This procedure is transparent in the example above.

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